6. NILPOTENT GROUPS

§6.1. The Ascending Central Series

The **centre** of a group G is

$$\mathbf{Z(G)} = \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

We can use this to define a whole series of 'centres', called the **ascending central series**.

W define $\mathbf{Z_i(G)}$ inductively by defining

$$Z_0(G) = 1 \text{ and}$$

$$Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G)).$$

So $Z_1(G) = Z(G)$ and $Z_{i+1}(G) = \{z \in \boldsymbol{G} \mid [z, g] \in Z_i(G)\}$ for all $i \geq 1$.

The **ascending central series** is thus:

$$1 \leq Z(G) \leq Z_2(G) \leq Z_3(G) \leq \ldots\ldots$$

We define G to be **nilpotent** if $Z_n(G) = G$ for some n, and the smallest such n is called the **nilpotency class**, or just the **class**. The trivial group is the only nilpotent group of class 0 and non-trivial abelian groups are nilpotent groups of class 1.

Example 1: The dihedral group of order 8 is nilpotent of class 2.
If $G = \langle A, B \mid A^4, B^2, [A, B] = A^2\rangle$ we have $Z(G) = \langle A^2\rangle$ and $Z_2(G) = G$.

Example 2: Generalizing example 1,
if $G = \mathbf{D}_{2^{n+1}} = \langle A^{2^n}, B^2, [A, B] = A^2 \rangle$
is the dihedral group of order 2^{n+1} then for $r < n$,
$Z_r(G) = \langle A^{2^{n-r}} \rangle$ and $Z_n(G) = G$.
Hence G is nilpotent of class n.

Theorem 1: Finite p-groups are nilpotent.
Proof: $Z_{r+1}(G)/Z_r(G) = Z(G/Z_r(G))$.
Since the centre of a non-trivial p-group is non-trivial,
$Z_r(G) < Z_{r+1}(G)$ unless $Z_r(G) = G$.

Theorem 2: Subgroups and quotient groups of a nilpotent group of class n are nilpotent and their nilpotency class is at most n.

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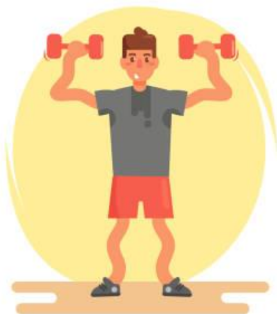

NILPOTENT

Proof:
Suppose $Z_n(G) = G$ and let $H \leq G$.
For each r, $Z_r(H) \geq Z_r(G) \cap H$ and so $Z_n(H) = H$.
Suppose now that H is normal in G.
For each r, $Z_r(G)H/H \leq Z_r(G/H)$ and so $Z_n(G/H) = G/H$.

Theorem 3:
Let G be a nilpotent group of class n and let $H \leq G$.
Let $H_0 \leq H_1 \leq H_2 \leq \ldots$ be defined by:

$$H_0 = H,$$
$$H_{i+1} = N_G(H_i) \text{ for all } i \geq 0.$$

Then $H_n = G$.
Proof: $1 = Z_0(G) \leq H_0$.
We prove by induction on r that $H_r \leq Z_r(G)$.
For $z \in Z_{r+1}(G)$ and $h \in H_r$, $[z, h] \in Z_r(G)$ and so
So $z^{-1}h^{-1}z \in H_rZ_r(G) \leq H_r$.
Hence $z \in N_G(H_r) = H_{r+1}$. Thus $Z_{r+1}(G) \leq H_{r+1}$.

Corollary 1: If G is nilpotent and $H < G$ then $H < N_G(H)$.
Proof: If $H = N_G(H)$ then H_r, as defined above, will be equal to H for all r.

Corollary 2: Every maximal subgroup of a nilpotent group is normal.

Corollary 3: Every maximal subgroup of a nilpotent group has finite, prime index.
Proof: If M is a maximal subgroup then G/M has no proper subgroups and so is isomorphic to $\mathbf{C}_p$ for some prime p.

These two corollaries generalise what we have proved for p-groups.

Theorem 4: A finite group is nilpotent if and only if it is a direct product of its Sylow subgroups.

Proof: Finite p-groups are nilpotent, and hence so is any direct product of p-groups.

Now suppose that G is a finite nilpotent group and let P be any Sylow subgroup of G.

Let $N = N_G(P)$. Suppose $N < G$.

Then $N < N_G(N)$, by Theorem 4 (Corollary 1).

Let $x \in N_G(N) - N$.

Hence $x^{-1}Px \le x^{-1}Nx = N$ and so $x^{-1}Px$ is a Sylow subgroup of N, as is P.

It follows that $y^{-1}x^{-1}Pxy = P$ for some $y \in N$ and so $xy \in N$.

But $y \in N$ and $x \notin N$, a contradiction.

It must be therefore be that $N = G$ and so $P \trianglelefteq G$.

G is therefore the direct product of its Sylow subgroups.

Corollary: A finite group is nilpotent if it has a unique Sylow subgroup for each of the primes dividing its order.

Proof: Left as an exercise.

§6.2. The Descending Central Series

If H, K are subgroups of a group G we define

$$\mathbf{[H, K]} = \langle [h, k] \mid h \in H, k \in K \rangle.$$

In particular, $G' = [G, G]$.

Recall that the higher commutators are defined by: $[x_1, x_2, \ldots, x_k] = [[x_1, x_2, \ldots, x_{k-1}], x_k]$. The **weight** of the commutator $[x_1, x_2, \ldots, x_k]$ is defined to be k.

Let $\Gamma_1(G) = G$ and $\Gamma_k(G) = \langle [x_1, x_2, \ldots, x_k \mid x_i \in G\rangle$. The **descending central series** is:

$$G = \Gamma_1(G) \geq \Gamma_2(G) \geq \ldots$$

Note that $\Gamma_1(G) = G'$ and $\Gamma_{k+1}(G) \leq [\Gamma_k(G), G]$.

It is not obvious that the inclusion works the other way round since $\Gamma_{k+1}(G)$ is generated by all commutators of the form $[x, y]$ where $y \in G$ and x is a commutator of weight k while $[\Gamma_k(G), G]$ is generated by commutators of the form $[x, y]$ where $y \in G$ and x is a *product* of commutators of weight k as well as inverses of these.

Theorem 5: For all k, $\Gamma_{k+1}(G) = [\Gamma_k(G), G]$.
Proof: Clearly $\Gamma_{k+1}(G) \leq [\Gamma_k(G), G]$.
We have to prove that if $x_1, x_2, \ldots, x_n \in \Gamma_k(G)$ and $y \in G$ then $[x_1x_2, \ldots x_n, y] \in \Gamma_{k+1}(G)$.

Now we have the identity:

$$[xy, z] = [x, z][x, z, y][y, z]$$

So we can prove by induction on n that $[x_1x_2, \ldots x_n, y]$ is a product of commutators of weight $k + 1$ and hence lies in $\Gamma_{k+1}(G)$.

§6.3. Nilpotent Groups of Class 2

Nilpotent groups of class 2 have many properties in common with abelian groups. In an abelian group we have $(xy)^n = x^n y^n$ for all x, y. If the group has class 2 there's a similar, but slightly more complicated result.

Theorem 6: If G is nilpotent of class 2 then

$$(xy)^n = x^n y^n [y, x]^{n(n-1)/2}.$$

Proof: We prove this by induction on n.
For $n = 1$ it is obvious.
Suppose it's true for n.
Then $(xy)^{n+1} = (xy)^n(xy) = x^n y^n [y, x]^{1n(n-1)/2} xy$.
Since $[y, x] \in Z(G)$ we can write this as $x^n y^n xy[y, x]^{n(n-1)/2}$.
Now $yx = xy[y, x]$ so that each time we bring an x to the left, past a y, we introduce a factor of $[y, x]$. These factors can be moved together with all the others, at the end of the expression.

Hence $y^n x = xy^n[y, x]^n$ and so

$$(xy)^{n+1} = x^{n+1} y^{n+1} [y, x]^{1/2\, n(n-1) + n}$$
$$= x^{n+1} y^{n+1} [y, x]^{1/2 n(n+1)}.$$

So it's true for all n.

Theorem 7: In a nilpotent group of class 2 conjugates commute with one another.

Proof: Let $x, y \in G$. Since $x^{-1}yx = y[y, x]$ and $[y, x] \in Z(G)$, $x^{-1}yx$ commutes with y.

Theorem 8: If G is nilpotent of class 2 then

$$\tau(G) = \{g \in G \mid g^n = 1 \text{ for some } n > 0\}$$

is a normal subgroup of G.

Proof: Let $x, y \in \tau(G)$. Then $x^m = y^n = 1$ for some m, n.
Since $[y, x] = y^{-1}(x^{-1}yx)$ it follows that
$[y, x]^n = y^{-n}(x^{-1}yx)^n = y^{-n}x^{-1}y^n x = 1$.
Hence $(xy)^{2mn} = x^{2mn}y^{2mn}[y, x]^{mn(2mn+1)} = 1$.
The normality is obvious.

§6.4. Verbally Abelian Groups

A group G is **verbally abelian** if there exists a word $W(x, y)$ in two variables such that $(G, *)$ is an abelian group under the operation $x * y = W(x, y)$.

When a group is verbally abelian we have two group structures on the same set. Suppose G is the original group and G_* is the abelian group on the set G. Then subgroups of G are subgroups of G_*. The order of elements is the same in both groups and any automorphism of G is automatically an automorphism of G_*.

Theorem 9: Suppose G is a nilpotent group of class 2 and n is an odd integer such that $g^n = 1$ for all $g \in G'$. Then G is verbally abelian.

Proof: Let G be nilpotent of class 2 and suppose that n is odd and $g^n = 1$ for all $g \in G'$.

It is easily checked that for all k, (G, $*$) is a group under the operation $x * y = xy[x, y]^k$. (This is left as a routine exercise.)

Suppose $k = \dfrac{n-1}{2}$.

Then $x * y = xy[x, y]^k$ and $y * x = yx[y, x]^k$.

Since $yx = xy[y, x]$, we have $y * x = xy[y, x]^{k+1}$

$$= xy[x, y]^{-k-1}.$$

But $1 = [x, y]^n = [x, y]^{2k+1}$ so $[y, x]^k = [x, y]^{-k-1}$.

Hence (G, $*$) is abelian.

EXERCISES FOR CHAPTER 6

Exercise 1: For each of the following statements determine whether it is true or false.

(1) Abelian groups are nilpotent.

(2) Every nilpotent group is soluble.

(3) Every metacyclic group is nilpotent.

(4) If $G' \leq Z(G)$ and $G' \cong \mathbf{S}_5$ then $(xy)^{16} = x^{16}y^{16}$ for all $x, y \in G$.

(5) If G is nilpotent of class 2 then $(xy)^n = x^n y^n$ for all $x, y \in G$.

(6) If G is nilpotent then it is a direct product of p-groups.

(7) Dihedral groups of order $4k$ where k is odd, are verbally abelian.

(8) There are some nilpotent groups of class 2 for which $x \bullet y = xy[y, x]^{13}$ is not a group word.

Exercise 2: Prove that all groups of order 6125 are nilpotent.

Exercise 3: Prove that if G is a nilpotent group of class 2 and a binary operation $\bullet$ is defined by:

$$x \bullet y = xy[x, y]^k \text{ for some integer } k$$

then $(G, \bullet)$ is a group.

SOLUTIONS FOR CHAPTER 6

Exercise 1:
(1) TRUE
(2) TRUE
(3) FALSE: $\mathbf{S}_3$ is metacyclic but not nilpotent]
(4) TRUE: $(xy)^{16} = x^{16}y^{16}[y, x]^{16.15/2} = x^{16}y^{16}[y, x]^{120}$.
If $G' \cong \mathbf{S}_5$ then $[y, x]^{120} = 1$.
(5) FALSE: $\mathbf{D}_8$ is nilpotent of class 2.
However $(xy)^2 = x^2y^2[y, x]$.
[Groups where $(xy)^2 = x^2y^2$ must be abelian.]
(6) FALSE: This is only true for *finite* nilpotent groups.
$\langle A, B \mid B^{-1}AB = A^{-1}\rangle$ is nilpotent of class 2 but has no Sylow subgroups.
(7) TRUE: $G \cong \langle A, B \mid A^{2k}, B^2, B^{-1}AB\rangle$.
$G' = \langle A^2\rangle$ which has odd order k. Then by Theorem 9, G is verbally abelian.
(8) FALSE: (G, •) is always a group. The restrictions in Theorem 9 are only needed to make (G, •) abelian.

Exercise 2: $6125 = 5^3.7^2$
The number of Sylow 5-subgroups is $\equiv 1 (\text{mod } 5)$ and divides 49, and so must be 1.
The number of Sylow 7-subgroups is $\equiv 1 (\text{mod } 7)$ and divides 125, and so must be 1.
Hence a group, G, of order 6125 must have unique Sylow 5-subgroup, H, and a unique Sylow 7-subgroup, K.

These must be normal in G and so, by the corollary to Theorem 4, G is nilpotent.

Exercise 3:

Associativity: $(x \bullet y) \bullet z = xy[x, y]^k z\, [xy[x, y]^k, z]^k$

$$= xyz\, [x, y]^k ([x, z][y, z])^k$$

$$= xyz\, [x, y]^k [x, z]^k [y, z]^k$$

$$= x \bullet (y \bullet z).$$

Identity: $x \bullet 1 = x1[x, 1]^k = x$ for all x.

Inverse: $x \bullet x^{-1} = xx^{-1}[x, x^{-1}] = 1$.